Lattice properties of acyclic pipe dreams

Propriétés de treillis des arrangements de tuyaux acycliques

Noémie Cartier

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- réflexive

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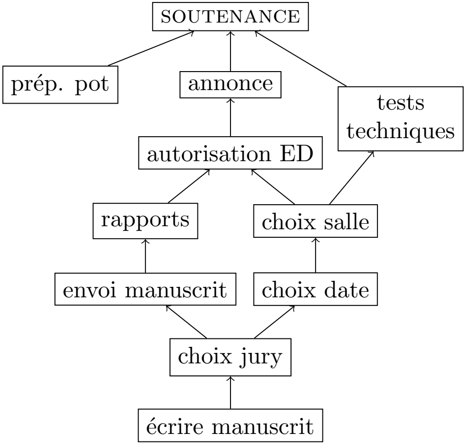

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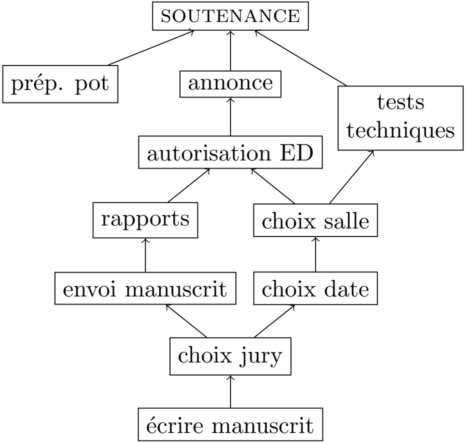
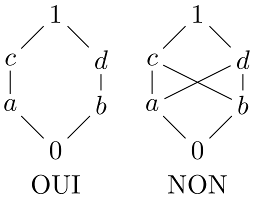
Extension linéaire : ordre total compatible avec l'ordre partiel



Qu'est-ce qu'un treillis ?

Un poset $(X, \leqslant)$ est un **treillis** si toute paire $a, b \in X$ possède :

- un **join** ou borne supérieure $a \vee b$;
- un **meet** ou borne inférieure $a \wedge b$.



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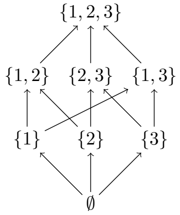
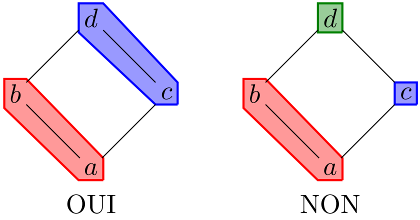
Exemples classiques :

- le **treillis booléen** $(\mathcal{P}(A), \subseteq)$: union et intersection;
- l'**ordre de divisibilité** sur les entiers positifs : PGCD et PPCM.



$\equiv$ relation d'équivalence sur X treillis est une **congruence de treillis** si :

$$\begin{matrix} x \equiv x' \\ y \equiv y' \end{matrix} \iff \begin{matrix} x \vee y \equiv x' \vee y' \\ x \wedge y \equiv x' \wedge y' \end{matrix}$$



$a \vee c = c \not\equiv d = b \vee c$

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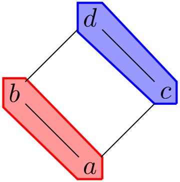

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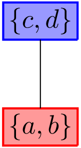
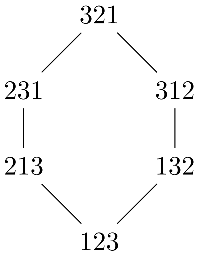
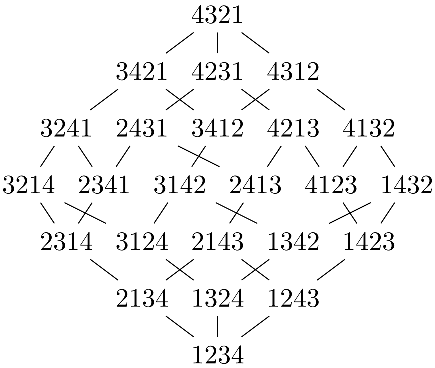


$\Rightarrow$ **quotient de treillis** $X/\equiv$: poset induit par $\leqslant$ sur les classes d'équivalence de $\equiv$

Ordre faible sur les permutations :





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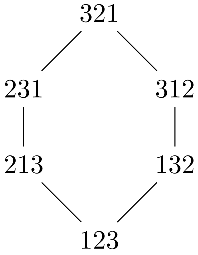

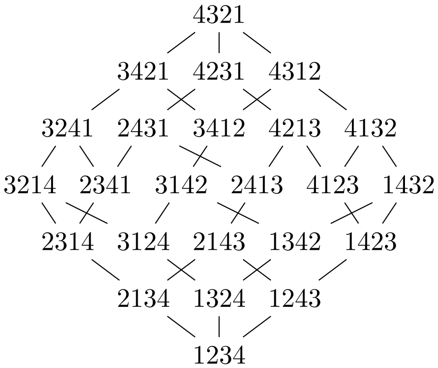

Défini par l'inclusion sur les **ensembles d'inversions** :

$\mathrm{inv}(\omega) := \{i < j \text{ and } \omega^{-1}(i) > \omega^{-1}(j)\} \quad \rightarrow (1, 2) \in \mathrm{inv}(24135)$

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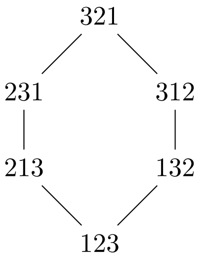

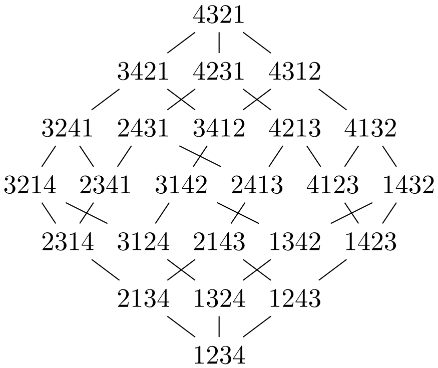

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L'ordre faible sur $\mathfrak{S}_n$ est un **treillis** (Guilbaud–Rosenstiehl, '63).

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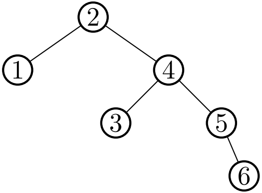

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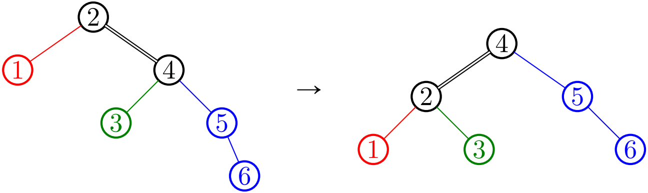

Opération d'équilibrage : la **rotation** (Adelson-Velsky–Landis, '62)

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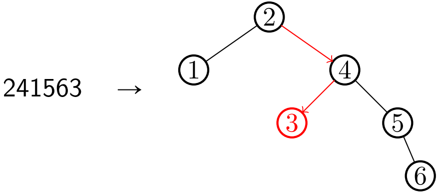

Des permutations aux arbres binaires : l'**insertion dans un ABR**

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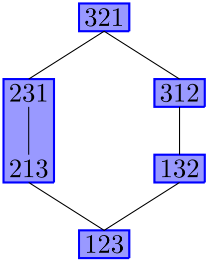

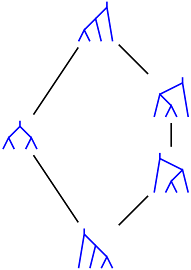

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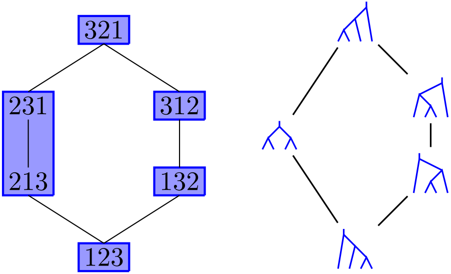

L'algorithme d'insertion dans les ABR définit un **morphisme de treillis** (Hivert–Novelli–Thibon, '05).

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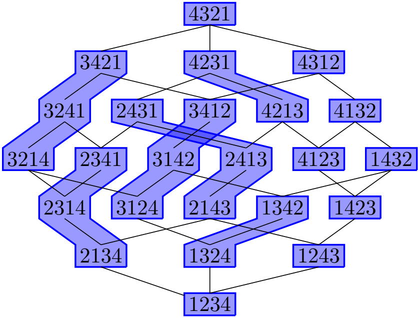

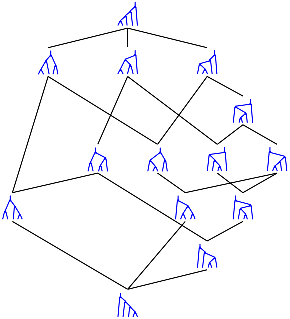

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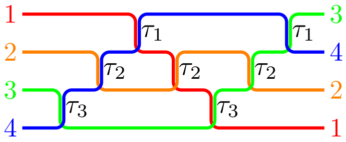

Sorting network $\leftrightarrow$ simple reflections product

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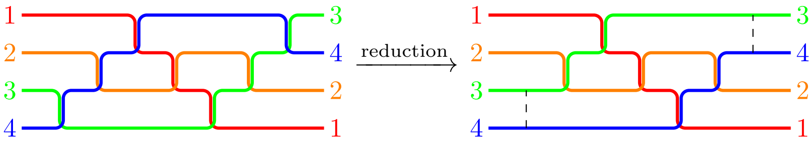

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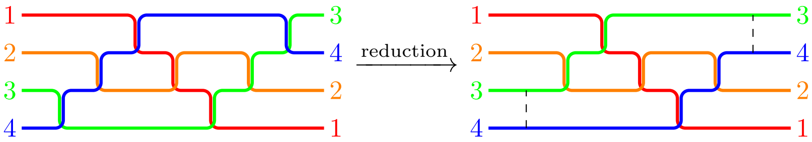

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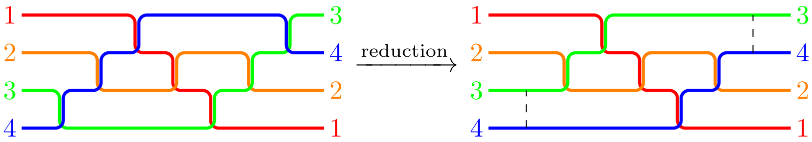

- $\pi \leqslant \omega$ iff $\omega = \pi\sigma$ and $\ell(\omega) = \ell(\pi) + \ell(\sigma)$: π is a **prefix** of ω
- if $\pi \leqslant \omega$ then any reduced expression of ω has a reduced expression of π as a **subword**

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- facets: complements of reduced subwords representing ω

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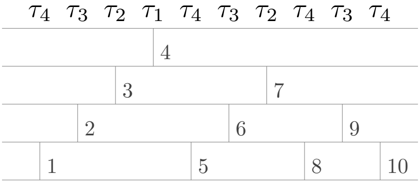

Facet $\{1, 2, 3, 8, 9\}$ of SC$(\tau_4\tau_3\tau_2\tau_1\tau_4\tau_3\tau_2\tau_4\tau_3\tau_4, 25143)$

Fix Q word on S, $\omega \in \mathfrak{S}_n$
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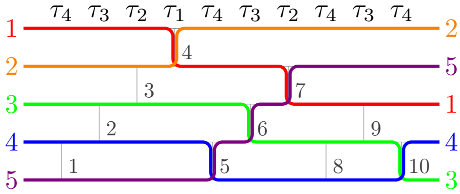

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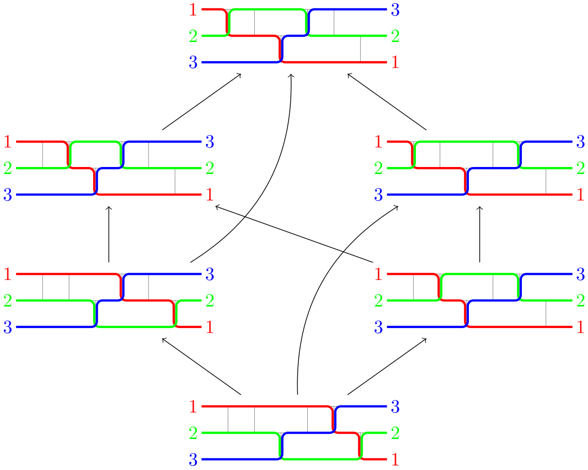

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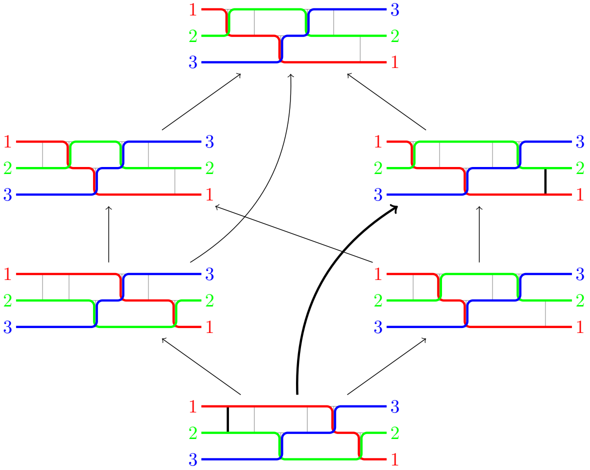
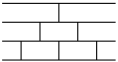
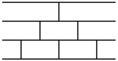
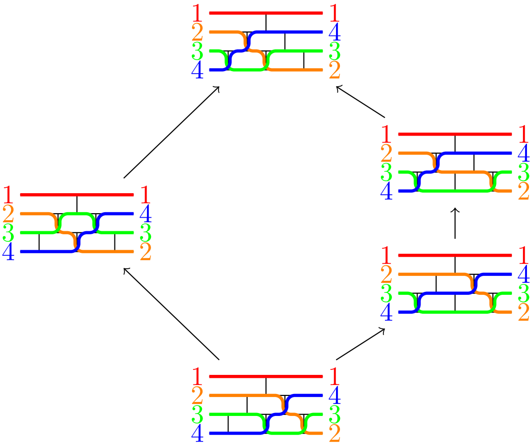
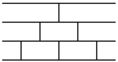


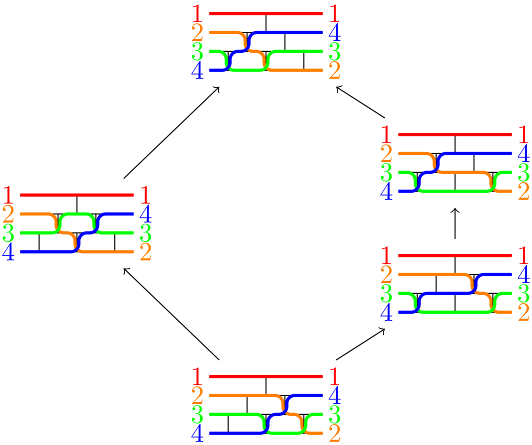
Q: triangular word  and $\omega = 1\, n\, (n-1)\, \dots\, 2$

A very special case

Q: triangular word  and $\omega = 1\, n\, (n-1)\, \ldots\, 2$



Q: triangular word  and $\omega = 1\, n\, (n-1)\, \ldots\, 2$



$\Rightarrow$ this is the Tamari lattice!

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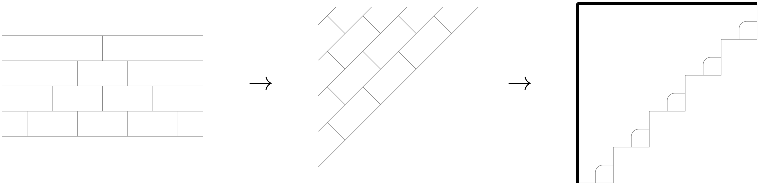

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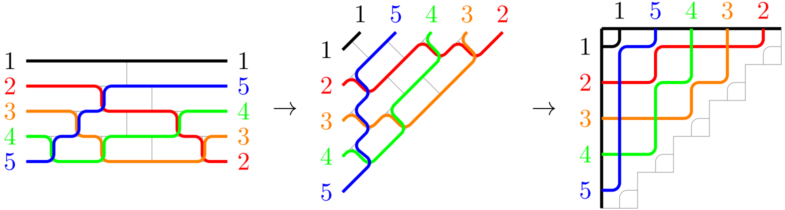

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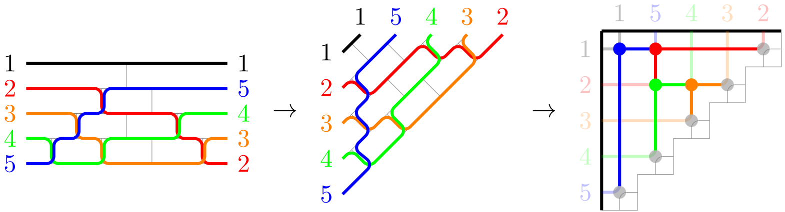

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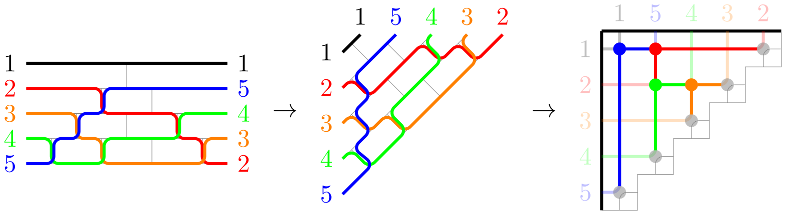

A binary tree appears on the pipe dream → bijection

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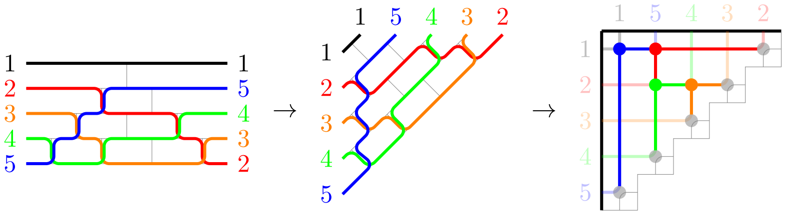

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Tree rotations ≡ flips → lattice isomorphism (Woo, 2004)

Can we find other lattice quotients of parts of the weak order with pipe dreams?

First extension: choose any exit permutation ω.

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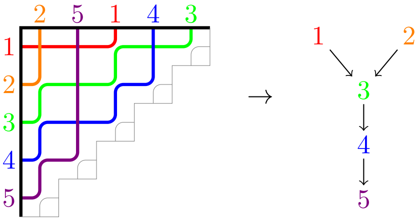
Contact graph:

- vertices: pipes
- edges: from a to b if a  b appears in the picture



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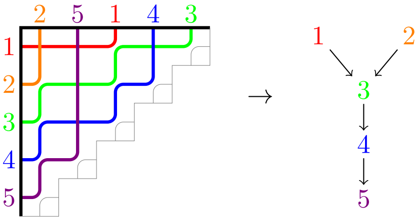

Acyclic contact graph $\iff$ vertex of **brick polytope** (Pilaud–Santos, '12)

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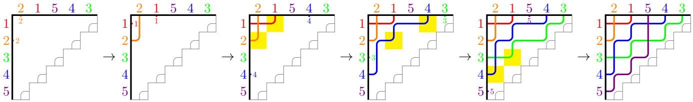
Theorem (Bergeron–C.–Ceballos–Pilaud)

For any $\omega \in \mathfrak{S}_n$, the ascending flip graph on $\Sigma(\omega)$ is a ***lattice quotient*** *of the weak order interval $[\mathrm{id}, \omega]$.*
The map $\mathrm{Ins}_\omega : [\mathrm{id}, \omega] \mapsto \Sigma(\omega)$ is a ***lattice morphism***.

Two algorithms to compute $\mathrm{Ins}_\omega(\pi)$: (for $\omega = 21543$ and $\pi = 21435$)

- insertion algorithm (pipe by pipe)



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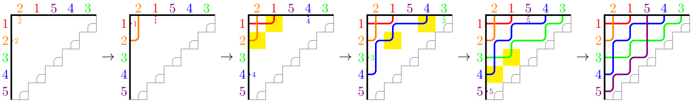
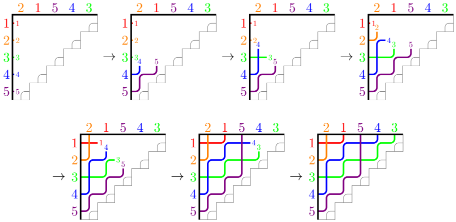
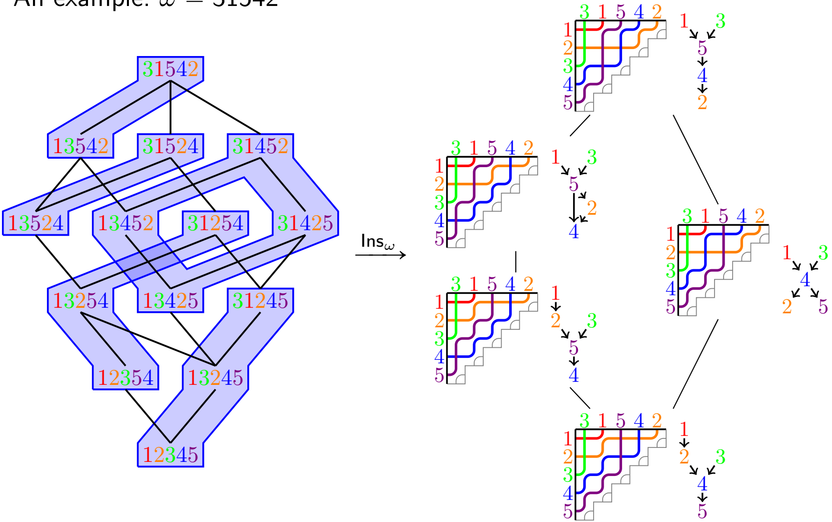


- sweeping algorithm (cell by cell)



An example: $\omega = 31542$



Second extension: other sorting networks

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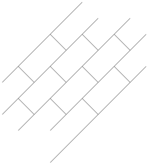
alternating sorting networks ↔ alternating shapes



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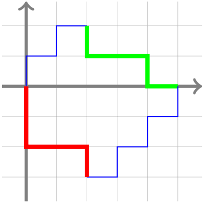

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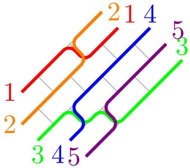

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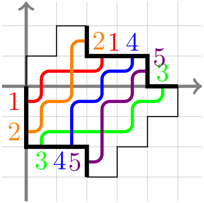

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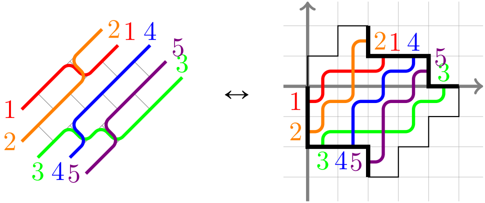

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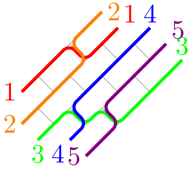

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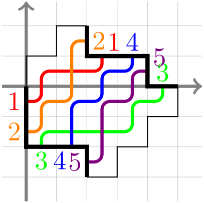

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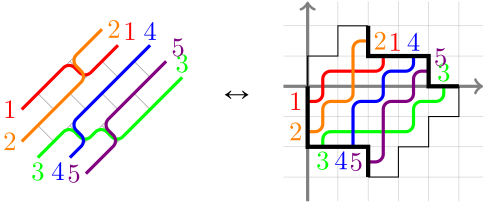

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Restrictions:

- $\Sigma_F(\omega)$ contains **strongly acyclic** pipe dreams
- order on $\Sigma_F(\omega)$: **acyclic order** (weaker than flip order)

Theorem (C.)

For any alternating shape F and $\omega \in \mathfrak{S}_n$ sortable on F, the acyclic order on $\Sigma_F(\omega)$ is a ***lattice quotient*** *of the weak order interval $[\mathrm{id}, \omega]$.*
The map $\mathrm{Ins}_{F,\omega} : [\mathrm{id}, \omega] \mapsto \Sigma_F(\omega)$ is a ***lattice morphism****.*

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Acyclic order $\leftrightarrow$ skeleton of (part of) the Brick polyhedron

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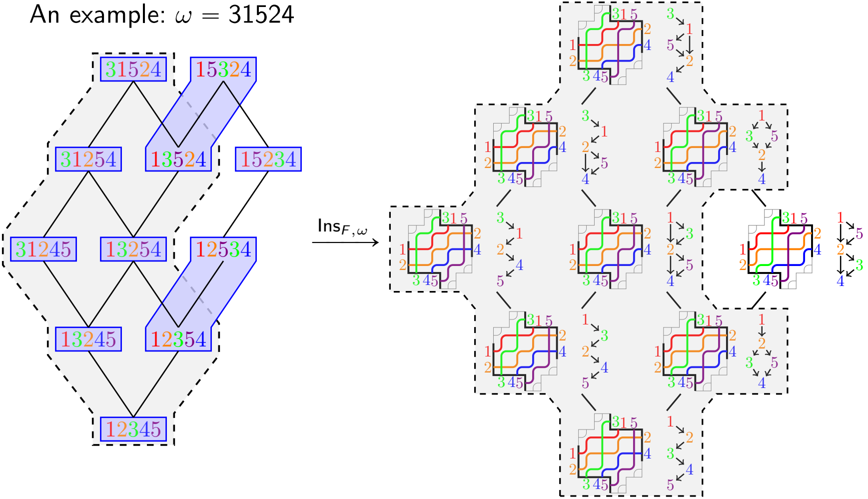
Acyclic order $\leftrightarrow$ skeleton of (part of) the Brick polyhedron

Theorem (C.)

*If the maximal permutation $\omega_0 = n\,(n-1)\,\dots 2\,1$ is sortable on F, then any linear extension of a pipe dream on F with exit permutation ω is in $[\mathrm{id}, \omega]$, and **all acyclic pipe dreams are strongly acyclic**.*

An example: $\omega = 31524$



Further generalization: Coxeter groups

symmetric group $\mathfrak{S}_n$	Coxeter group W
simple transpositions	simple reflections
weak order	
subword complexes	
pair of pipes	root in Φ
$P^{\#}$	root cone
$\pi \in \mathrm{lin}(P)$	root conf. $\subseteq \pi(\Phi^+)$

Theorem (BCCP)

For any word Q on S and $w \in W$ sortable on Q, the map $\mathrm{Ins}_{Q,w}$ is ***well-defined*** *on the weak order interval $[e, w]$.*

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If the Demazure product of Q is w_0, then for any $w \in W$ the application $\mathrm{Ins}_Q(w, \cdot)$ is ***surjective on acyclic facets*** *of $\mathrm{SC}(Q, w)$.*

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Conjecture

If Q is an alternating word on S and $w \in W$ is sortable on Q, then the application $\mathrm{Ins}_{Q,w} : [e, w] \mapsto \mathrm{SC}(Q, w)$ is a **lattice morphism** from the weak order on $[e, w]$ to the Brick polyhedron of $\mathrm{SC}(Q, w)$.

Thank you for your attention!

Merci pour votre attention !