Lattice Properties of Acyclic Pipe Dreams

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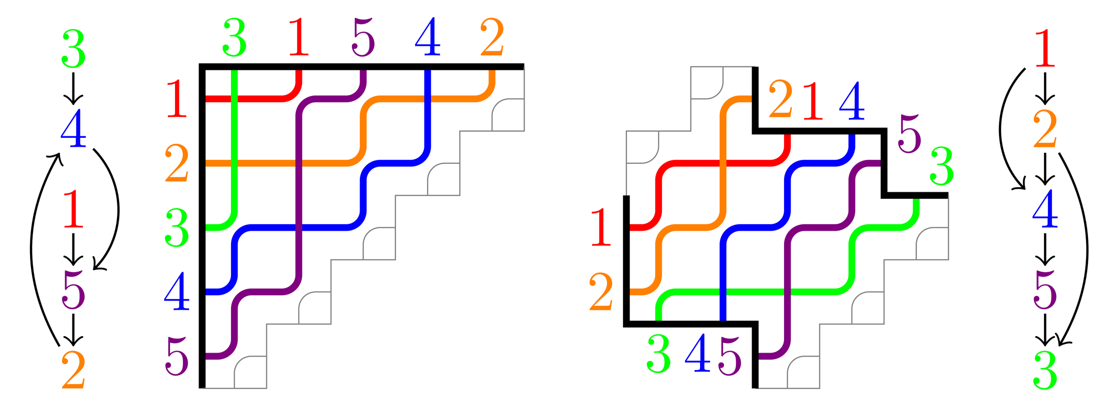
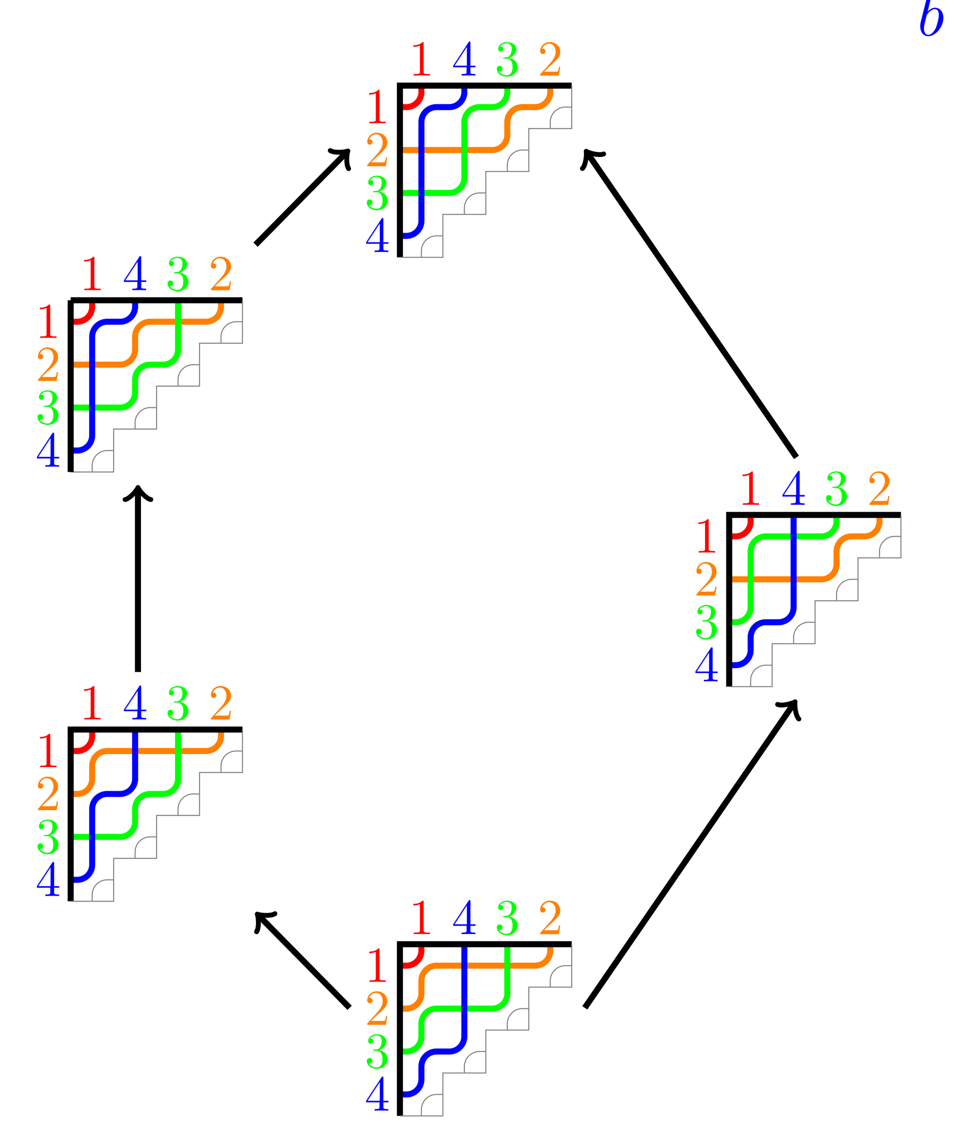
1. Pipe dreams

pipe dream: filling of shape with + and ⌐┘
reduced: no pair of pipes cross twice



contact graph $P^\#$: edge $a \to b$ if a ⌐┘ b in P
P **acyclic**: $P^\#$ acyclic
$\operatorname{lin}(P)$: **linear extensions** of $P^\#$

Increasing flip: exchanges a ⌐┘ b and a + b



$\Sigma_F(\omega) =$ {reduced acyclic pipe dreams on F with exit permutation ω}

More details?

Triangular: arXiv:2303.11025[a]

General: in preparation

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[a] Joint work w. N. Bergeron, C. Ceballos and V. Pilaud.

2. Insertion of permutations

Thm. *F a shape, ω a permutation:*

- $(\operatorname{lin}(P))_{P \in \Sigma_F(\omega)}$ *are disjoint*
- *if* $\pi \leqslant \omega$, $\exists! P \in \Sigma_F(\omega) \mid \pi \in \operatorname{lin}(P)$

Def. *Inserting permutations:*

- $\operatorname{Ins}_{F,\omega} : [\mathrm{id}, \omega] \mapsto \Sigma_F(\omega)$ *such that* $\pi \in \operatorname{lin}_{F,\omega}(\operatorname{Ins}_{F,\omega}(\pi))$
- $\pi \equiv_{F,\omega} \pi'$ *if and only if* $\operatorname{Ins}_{F,\omega}(\pi) = \operatorname{Ins}_{F,\omega}(\pi')$

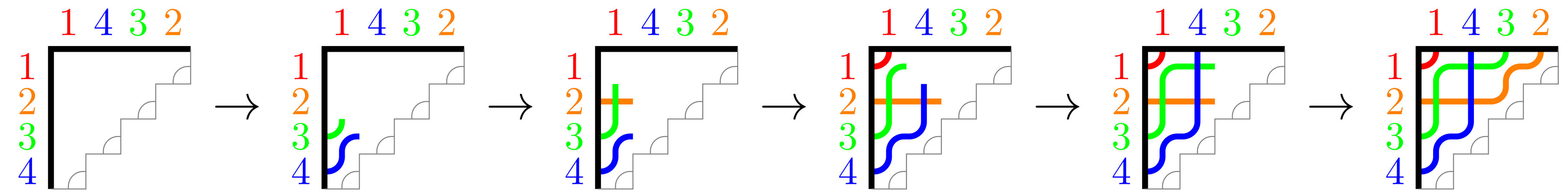
Prop. *If* $UabV \lessdot UbaV \leqslant \omega$:

- $\operatorname{Ins}_{F,\omega}(UabV) = \operatorname{Ins}_{F,\omega}(UbaV)$ *or*
- $\operatorname{Ins}_{F,\omega}(UabV) \to \operatorname{Ins}_{F,\omega}(UbaV)$ *is an increasing flip.*

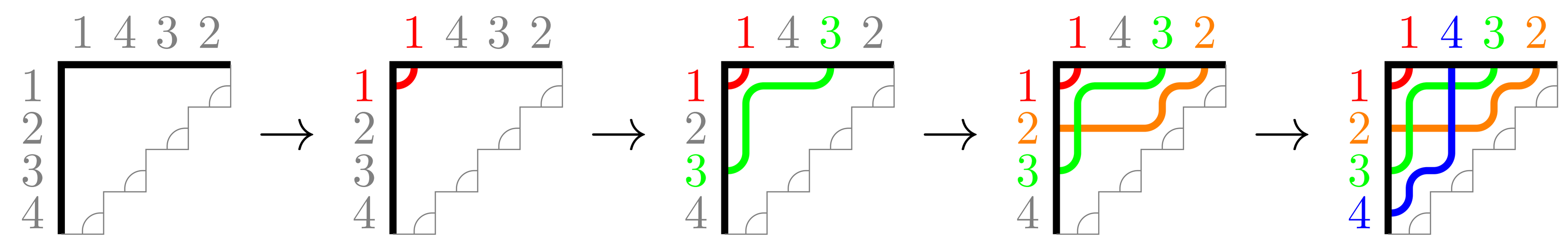
3. Two algorithms

Computing $\operatorname{Ins}_{T,1432}(1324)$:

Sweeping algorithm:



Insertion algorithm:

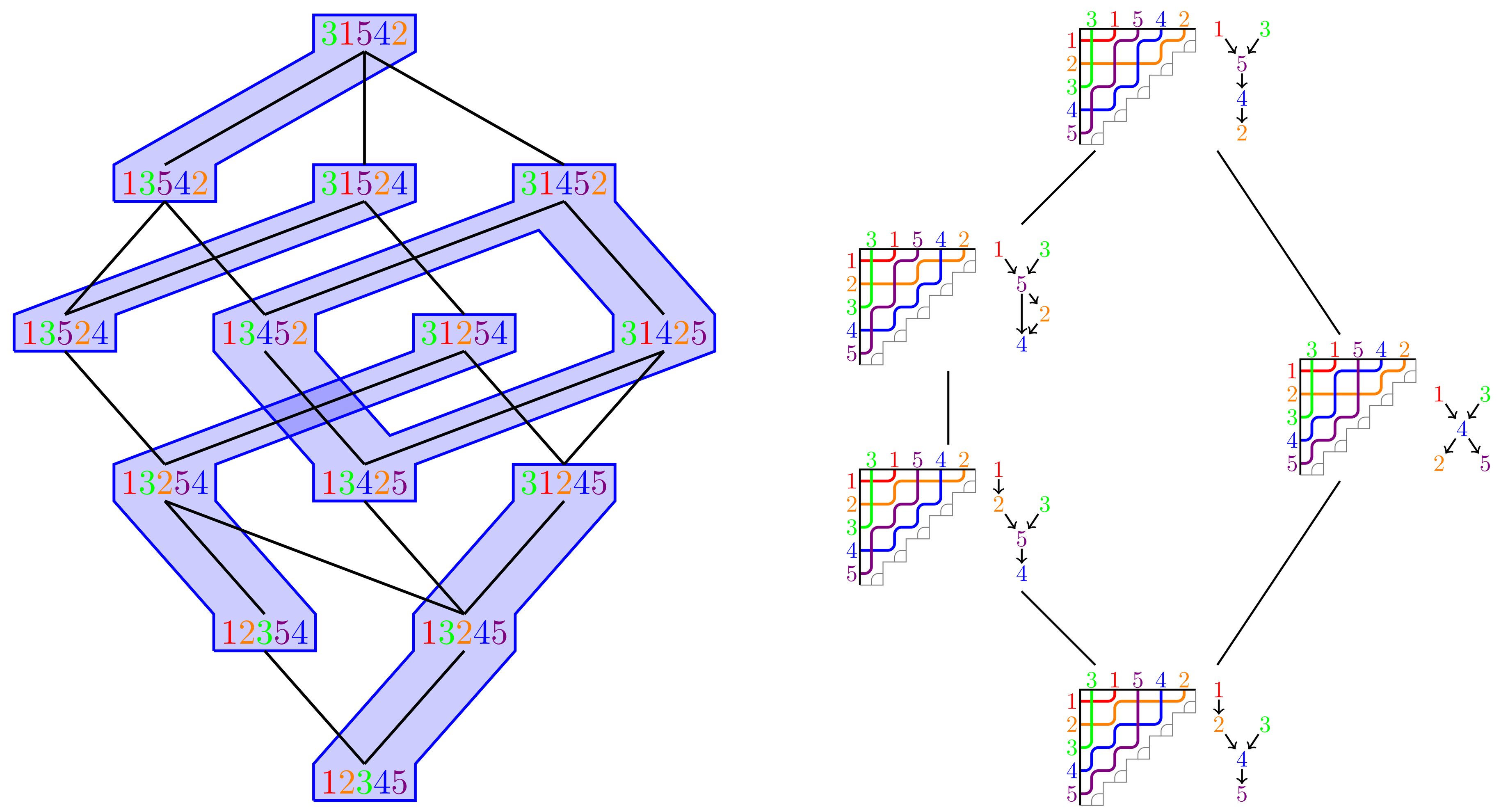


4. Triangular shapes

Thm.

- $\equiv_{T,\omega}$ ***lattice congruence***
- $\operatorname{Ins}_{T,\omega}$ ***lattice morphism***

weak order on $[\mathrm{id}, \omega]$ $\xrightarrow{\operatorname{Ins}_{T,\omega}}$ *increasing flip graph on* $\Sigma_T(\omega)$



Thm. $UabV \equiv_{T,\omega} UbaV$ *iff*

$$|\{k < a \mid k \lhd_\omega b\}| \leqslant \left| \left\{ k \lhd_\pi a \;\middle|\; \begin{array}{c} a < k < b \\ b \lhd_\omega k \lhd_\omega a \end{array} \right\} \right|$$

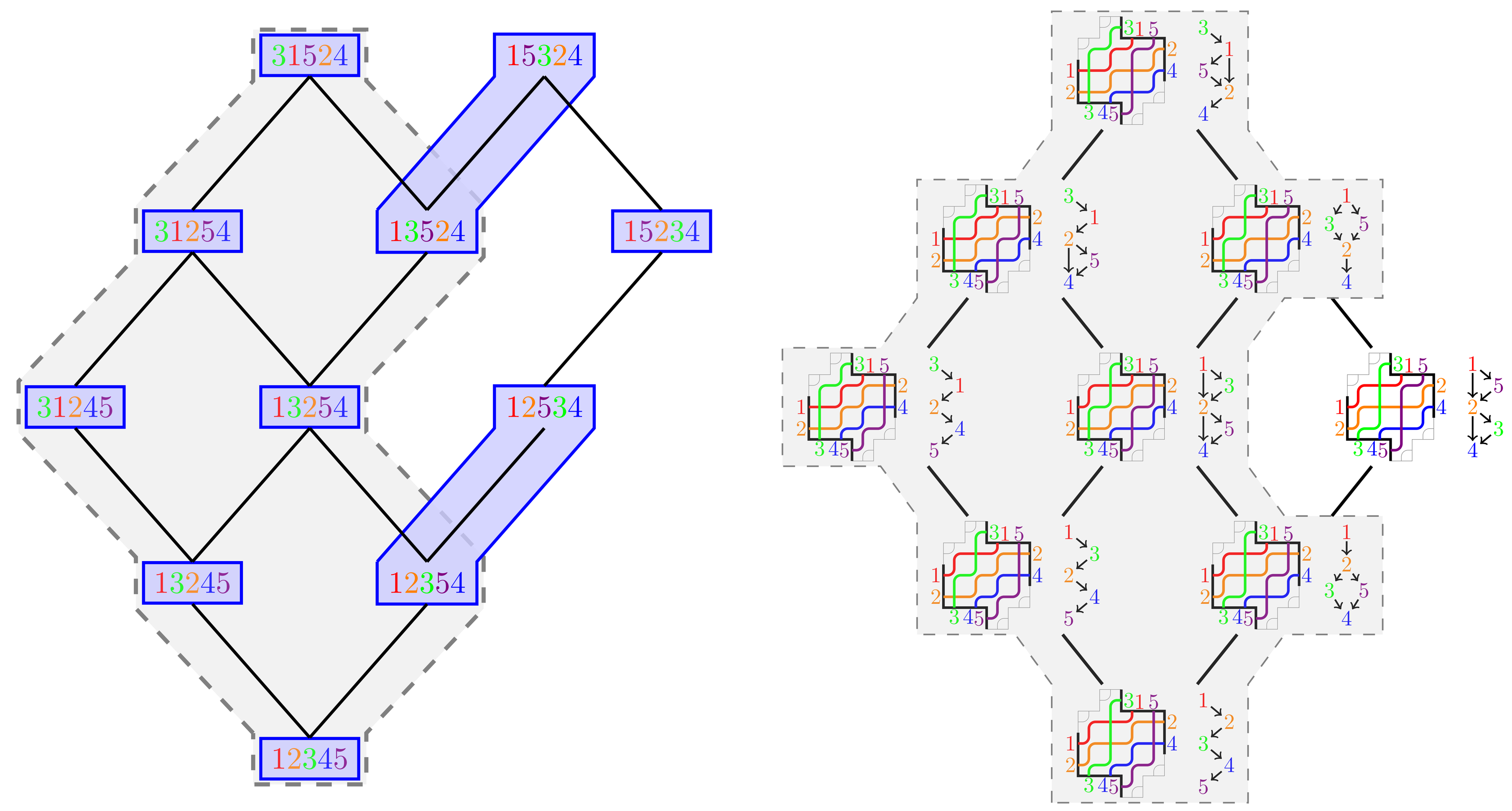
Generalizes the sylvester congruence from the weak order to the Tamari lattice

5. General shapes

Thm.

- $\equiv_{F,\omega}$ ***lattice congruence*** *on* $[\mathrm{id}, \omega]$
- $\operatorname{Ins}_{F,\omega}$ ***lattice morphism***

weak order on $[\mathrm{id}, \omega]$ $\xrightarrow{\operatorname{Ins}_{F,\omega}}$ *brick polyhedron skeleton on* $\operatorname{Ins}_{F,\omega}([\mathrm{id}, \omega])$



Thm. *If F sorts ω_0, then* $\forall \omega \in \mathfrak{S}_n$,

$$\operatorname{Ins}_{F,\omega}([\mathrm{id}, \omega]) = \Sigma_F(\omega)$$

Generalizes Cambrian congruences from the weak order to Cambrian lattices

6. Generalization to Coxeter groups

$\mathfrak{S}_n$	$\leftrightarrow$	Coxeter group
transp. $\tau_{i,i+1}$	$\leftrightarrow$	simple reflection
reduced pipe dreams	$\leftrightarrow$	subword complex
$\pi \in \operatorname{lin}(P)$	$\leftrightarrow$	root conf. $\subseteq \pi(\Phi^+)$

Thm. $\operatorname{Ins}_{Q,w}$ *is **well-defined** on* $[e, w]$.

Thm (Jahn & Stump '22). *Q sorts w_0* $\Rightarrow$ $\operatorname{Ins}_{Q,w}$ *is **surjective***

Conj. *Q is **alternating*** $\Rightarrow$

- $\equiv_{Q,w}$ *is a **lattice congruence***
- $\operatorname{Ins}_{Q,w}$ *is a **lattice morphism***