Lattice Properties of Acyclic Pipe Dreams

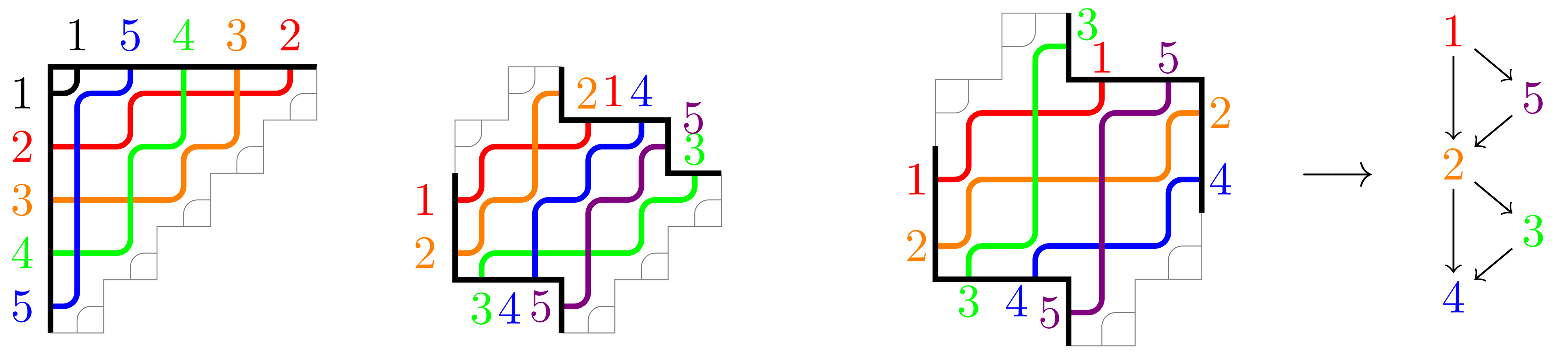
Noémie Cartier (LISN, Université Paris-Saclay)

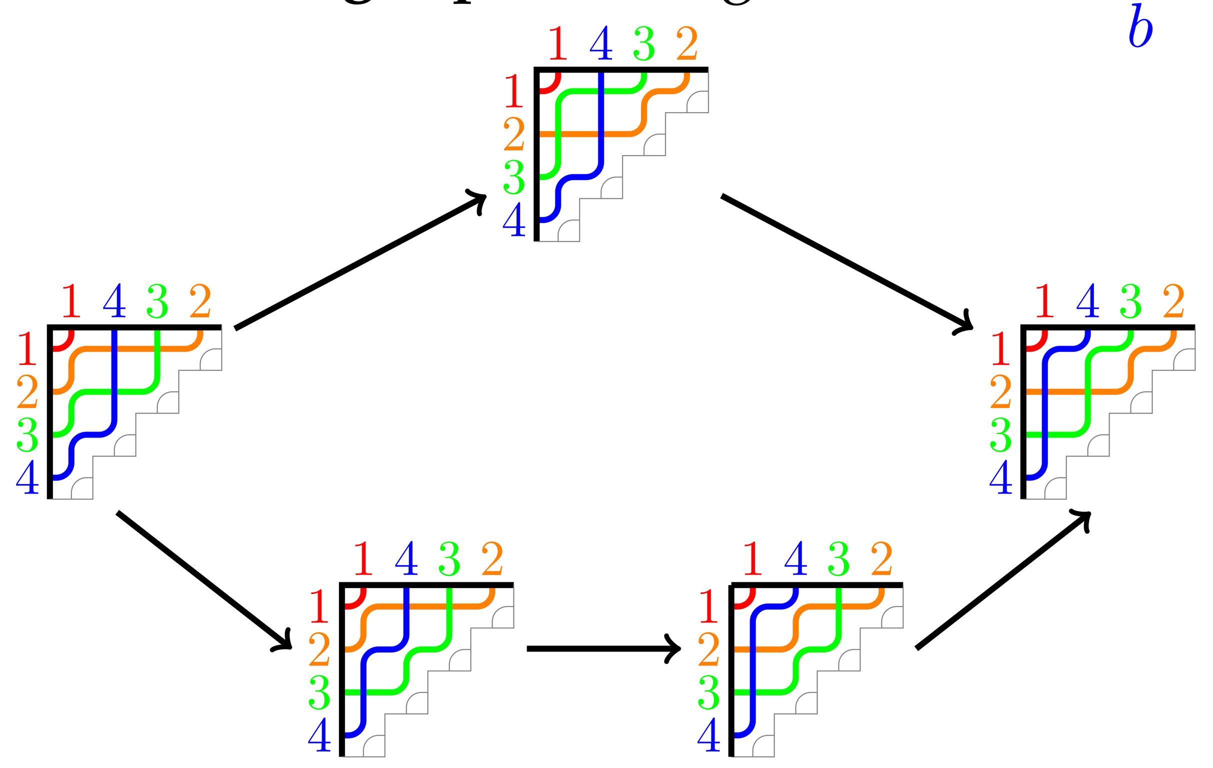
Pipe dreams

pipe dream: filling of shape with + and ⌐ (elbow)
reduced: no pair of pipes cross twice

contact graph $P^\#$: edge $a \to b$ if a ⌐ b in P
P **acyclic**: $P^\#$ acyclic

Increasing flip: exchanges a ⌐ b and a + b





$\Sigma_F(\omega) = \{$ reduced acyclic pipe dreams on F with exit permutation $\omega\}$

Insertion of permutations

Thm. *F a shape, ω a permutation:*
- $(\mathrm{lin}(P))_{P \in \Sigma_F(\omega)}$ *are disjoint*
- *if* $\pi \leqslant \omega$, $\exists! P \in \Sigma_F(\omega) \mid \pi \in \mathrm{lin}(P)$

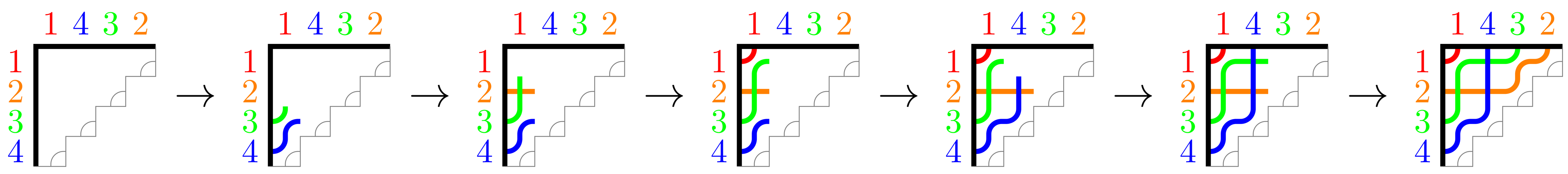
Def. *Inserting permutations:*
- $\mathrm{Ins}_{F,\omega} : [\mathrm{id}, \omega] \mapsto \Sigma_F(\omega)$ *such that* $\pi \in \mathrm{lin}_{F,\omega}(\mathrm{Ins}_{F,\omega}(\pi))$
- $\pi \equiv_{F,\omega} \pi'$ *iff* $\mathrm{Ins}_{F,\omega}(\pi) = \mathrm{Ins}_{F,\omega}(\pi')$

Prop. *If* $UabV \lessdot UbaV \leqslant \omega$:
- $\mathrm{Ins}_{F,\omega}(UabV) = \mathrm{Ins}_{F,\omega}(UbaV)$ *or*
- $\mathrm{Ins}_{F,\omega}(UabV) \to \mathrm{Ins}_{F,\omega}(UbaV)$ *is an increasing flip.*

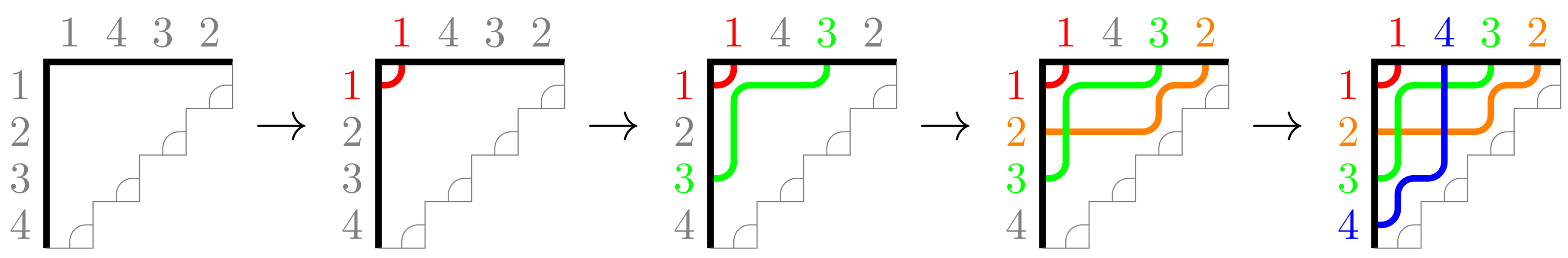
Two algorithms

Computing $\mathrm{Ins}_{T,1432}(1324)$:

Sweeping algorithm:



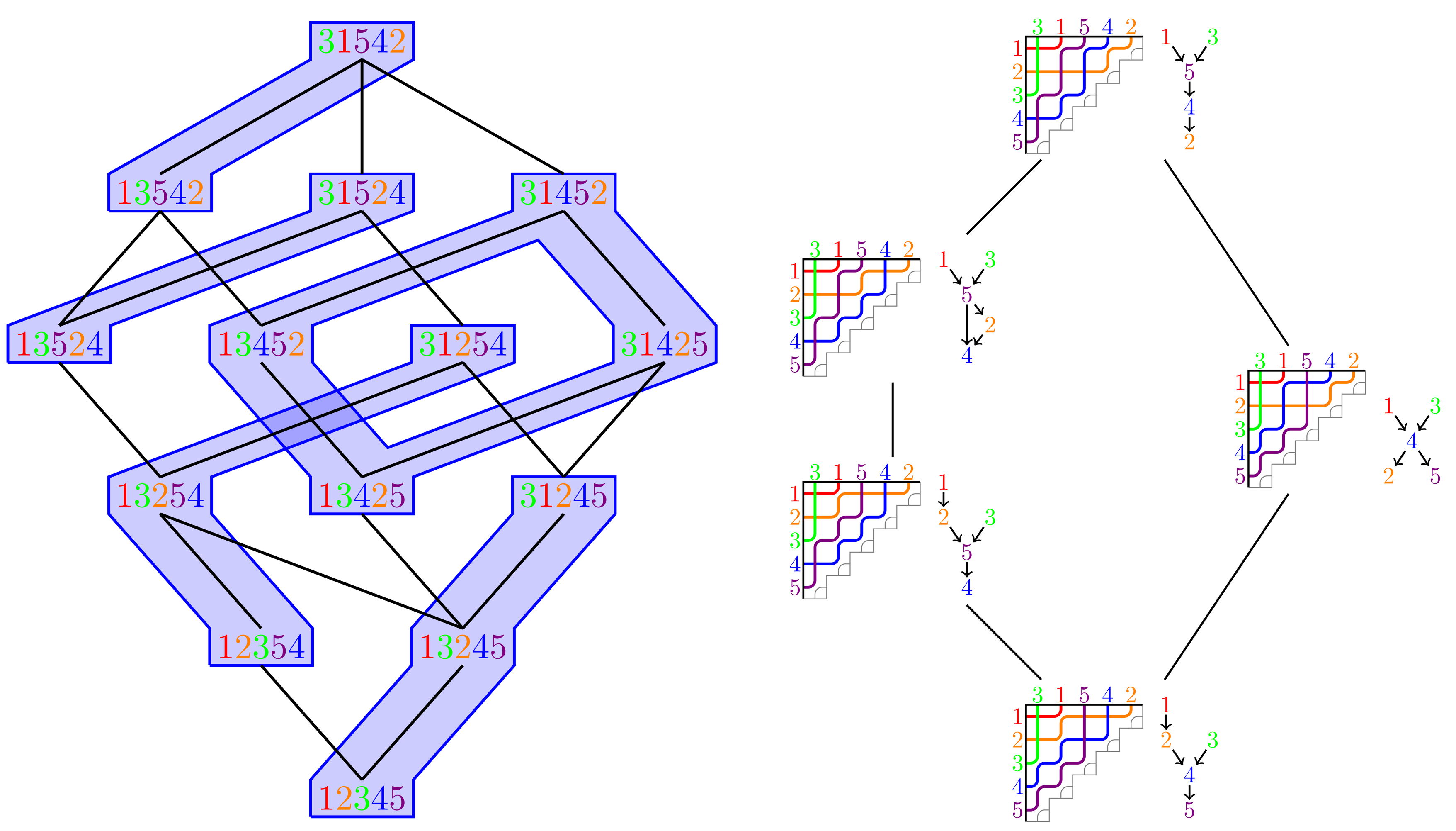
Insertion algorithm:



Triangular shapes

Thm.
- $\equiv_{T,\omega}$ ***lattice congruence***
- $\mathrm{Ins}_{T,\omega}$ ***lattice morphism***

weak order on $[\mathrm{id}, \omega]$ $\xrightarrow{\mathrm{Ins}_{T,\omega}}$ *increasing flip graph on* $\Sigma_T(\omega)$



Thm. $UabV \equiv_{T,\omega} UbaV$ *iff*

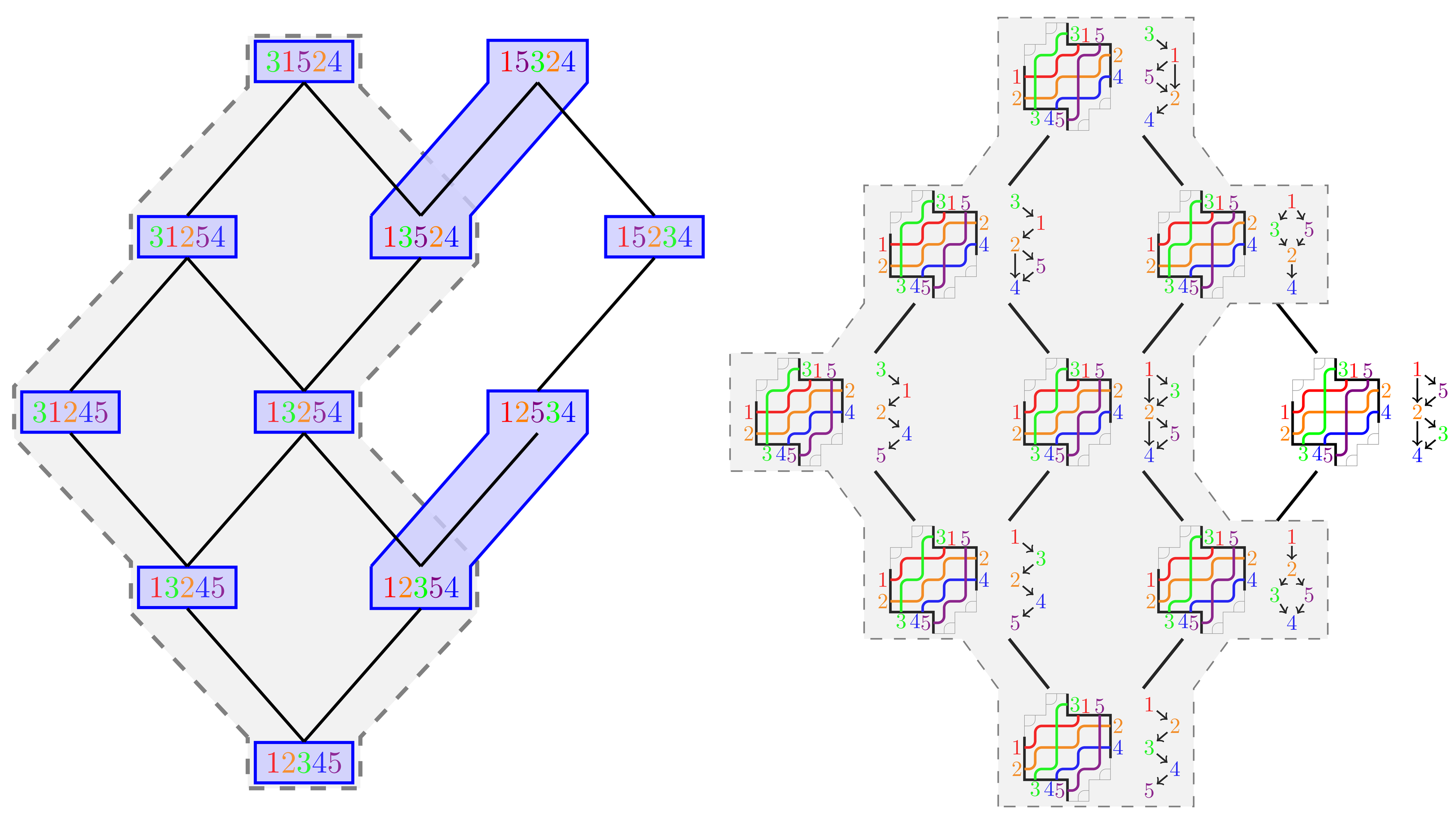
$$|\{k < a \mid k \lhd_\omega b\}| \leqslant \left| \left\{ k \lhd_\pi a \;\middle|\; \begin{array}{c} a < k < b \\ b \lhd_\omega k \lhd_\omega a \end{array} \right\} \right|$$

Generalizes the sylvester congruence from the weak order to the Tamari lattice

General shapes

Thm.
- $\equiv_{F,\omega}$ ***lattice congruence*** *on* $[\mathrm{id}, \omega]$
- $\mathrm{Ins}_{F,\omega}$ ***lattice morphism***

weak order on $[\mathrm{id}, \omega]$ $\xrightarrow{\mathrm{Ins}_{F,\omega}}$ *brick polyhedron skeleton on* $\mathrm{Ins}_{F,\omega}([\mathrm{id}, \omega])$



Thm. *If F sorts ω_0, then* $\forall \omega \in \mathfrak{S}_n$,

$$\mathrm{Ins}_{F,\omega}([\mathrm{id}, \omega]) = \Sigma_F(\omega)$$

Generalizes Cambrian congruences from the weak order to Cambrian lattices

Generalization to Coxeter groups

$\mathfrak{S}_n$	$\leftrightarrow$	Coxeter group
transp. $\tau_{i,i+1}$	$\leftrightarrow$	simple reflections
reduced pipe dreams	$\leftrightarrow$	subword complex
$\pi \in \mathrm{lin}(P)$	$\leftrightarrow$	root conf. $\subseteq \pi(\Phi^+)$

Thm. $\mathrm{Ins}_{Q,w}$ *is **well-defined** on* $[e, w]$

Thm (Jahn & Stump '22).
Q sorts w_0 $\Rightarrow$ $\mathrm{Ins}_{Q,w}$ ***surjective***

Conj. *Q **alternating*** $\Rightarrow$
- $\equiv_{Q,w}$ ***lattice congruence***
- $\mathrm{Ins}_{Q,w}$ ***lattice morphism***

More details?

Triangular: `arXiv:2303.11025`[a]
General: in preparation

Contact me:
noemie.cartier@lri.fr
https://www.lri.fr/~cartier/

[a]Joint work with N. Bergeron, C. Ceballos and V. Pilaud.