Lattice properties of acyclic pipe dreams

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What is a lattice ?

A poset $(X, \leqslant)$ is a **lattice** if and only if any pair $a, b \in X$ has :

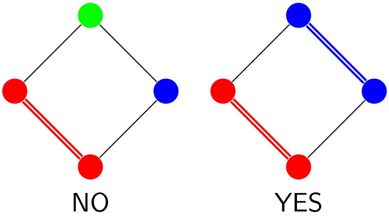
- a **join** or least upper bound $a \vee b$;
- a **meet** or greatest lower bound $a \wedge b$.

Some examples :

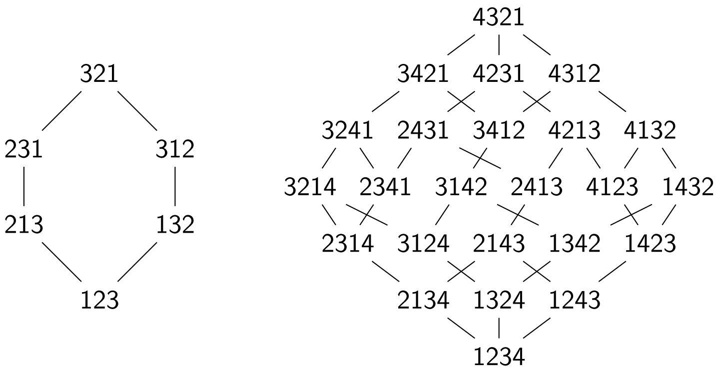
- the **boolean lattice** $(\mathcal{P}(A), \subseteq)$ on subsets of a set A, with the join given by $\cup$ and the meet by $\cap$;
- the **divisibility order** on positive integers.

For $(X, \leqslant, \vee, \wedge)$ a lattice and $\equiv$ an equivalence relation on X, we say that $\equiv$ defines a **lattice quotient** on X if for any $x, x', y, y' \in X$ such that $x \equiv x'$ and $y \equiv y'$:

- $x \vee y \equiv x' \vee y'$;
- $x \wedge y \equiv x' \wedge y'$.



Inversions of $\omega \in \mathfrak{S}_n$: $i < j$ and $\omega^{-1}(i) > \omega^{-1}(j)$ $\rightarrow (1, 2)$ in 24135

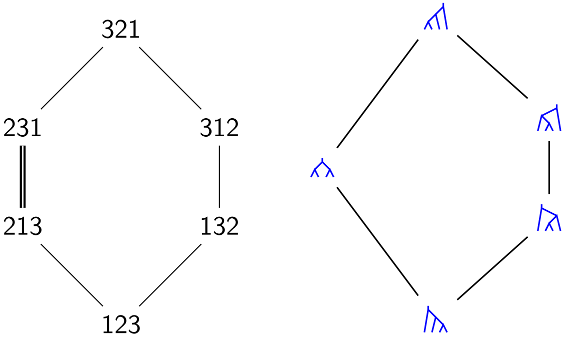


Right weak order on permutations : $\pi \leqslant \omega \iff \mathrm{inv}(\pi) \subseteq \mathrm{inv}(\omega)$

Theorem

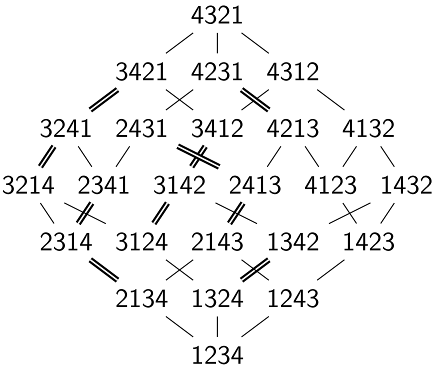
*The weak order on $\mathfrak{S}_n$ is a **lattice**.*

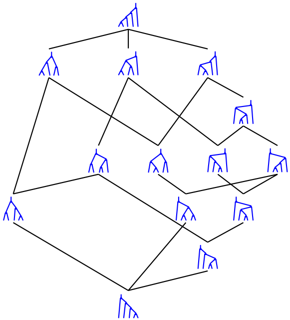
A classic lattice congruence of the weak order : the **Tamari lattice**



A **lattice morphism** : insertion into binary search trees.

A classic lattice congruence of the weak order : the **Tamari lattice**





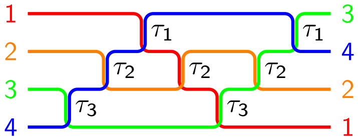
Covers of the right weak order :

$$UabV \lessdot UbaV$$
$$312456 \lessdot 314256$$

$\Leftrightarrow \quad \omega \lessdot \omega\tau_i$ with $\omega(i) < \omega(i+1)$

$\rightarrow$ importance of generating set $S = \{\tau_i = (i, i+1) \mid 1 \leqslant i < n\}$

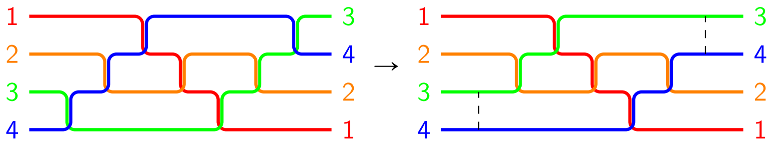
Sorting network $\leftrightarrow$ simple reflections product



Properties of words on S :

- minimal length for ω : $l(\omega) = |\,\mathrm{inv}(\omega)|$ (**reduced** words)
- $\pi \leqslant \omega$ iff $\omega = \pi\sigma$ and $l(\omega) = l(\pi) + l(\sigma)$: π is a **prefix** of ω
- if $\pi \leqslant \omega$ then any reduced expression of ω has a reduced expression of π as a **subword**

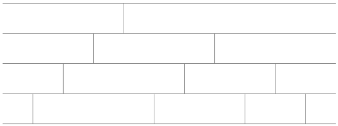
Reduction to minimal length :



Fix Q word on S, $\omega \in \mathfrak{S}_n$
$\mathrm{SC}(Q, \omega)$ the **subword complex** on Q representing ω :

- base set : indices of Q
- faces : complementaries of indices sets containing an expression of ω

An example :



Facet $\{1, 2, 3, 8, 9\}$ of $\mathrm{SC}(\tau_4\tau_3\tau_2\tau_1\tau_4\tau_3\tau_2\tau_4\tau_3\tau_4, 25143)$

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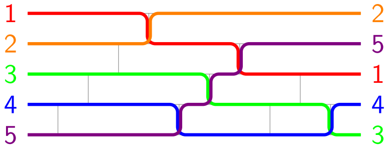

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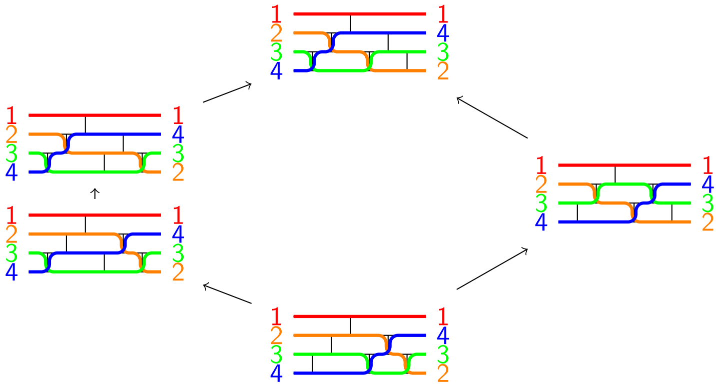
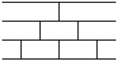
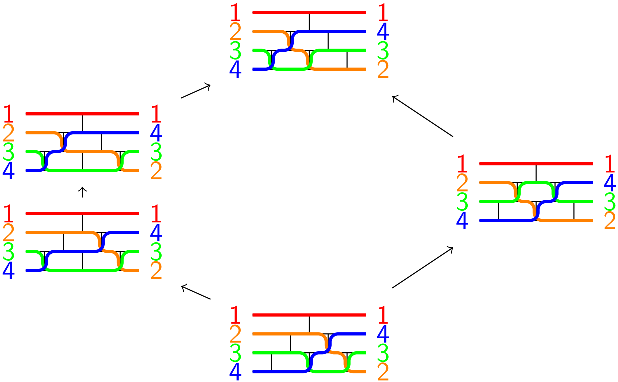
Structure given by **flips** : from one facet to another

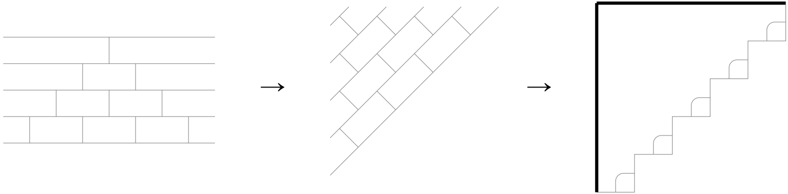


Q : triangular word  and $\omega = 1\ n\ (n-1)\ \ldots\ 2$



$\Rightarrow$ this is the Tamari lattice !

Why the Tamari lattice ?



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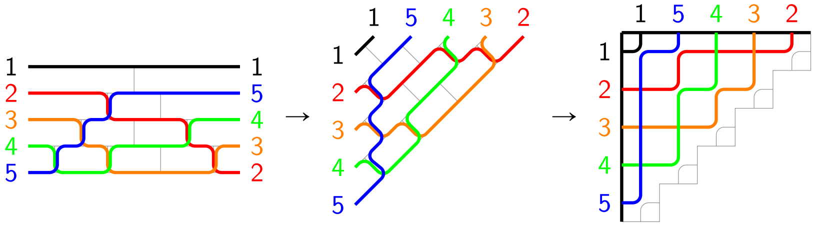

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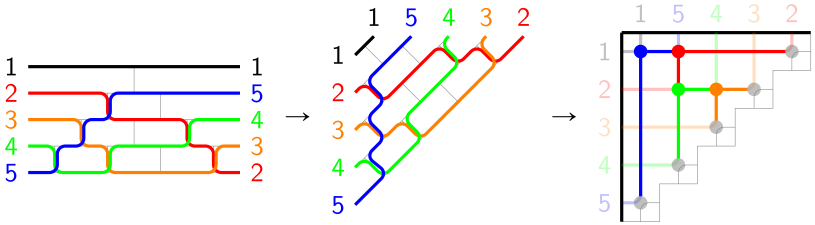
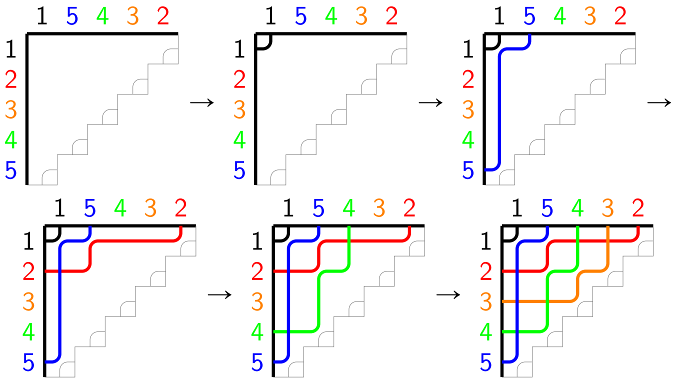


A binary tree appears on the pipe dream → bijection

Tree rotations ≡ flips → lattice isomorphism (Woo, 2004)

An equivalent to the insertion in binary search trees : **insertion algorithm on pipes**

Example : inserting permutation 15243



Reminder : the Tamari lattice is a **lattice quotient** of the weak order

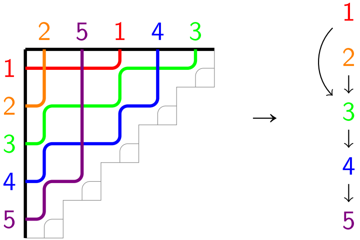
$\Rightarrow$ so is the flip order on this subword complex
$\Rightarrow$ lattice morphism : BST insertion $\iff$ pipes insertion

Can we find other pipe dream sets that are lattice quotients of parts of the weak order ?

First extension : choose any permutation for the exit.

Contact graph :

- vertices : pipes
- edges : from *a* to *b* if *a*  *b* appears in the picture



Why look at this ?

Acyclic contact graph $\Longleftrightarrow$ vertex of the **brick polytope**

First extension : choose any permutation ω for the exit.

Restriction : only consider the set of acyclic pipe dreams $\Pi(\omega)$
$\rightarrow$ from permutations to pipe dreams : contact graph extensions
$\rightarrow$ domain of the application : weak order interval $[\mathrm{id}, \omega]$
$\rightarrow$ name of the application : Ins_ω

Theorem (Pilaud)

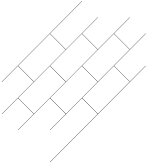
*For any $\omega \in \mathfrak{S}_n$, the map Ins_ω is a **lattice morphism** from the weak order interval $[\mathrm{id}, \omega]$ to the set of acyclic pipe dreams $\Pi(\omega)$ ordered by ascending flips.*

Two algorithms to compute the morphism :

- insertion algorithm (pipe by pipe)
- sweeping algorithm (cell by cell)

Second extension : other sorting networks

alternating sorting networks ↔ n-shapes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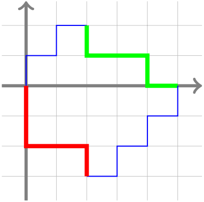

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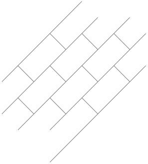

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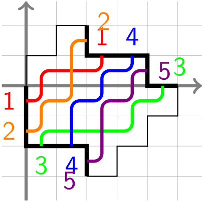


$\mathrm{Ins}_{F,\omega}$ is still well defined, BUT...

- some linear extensions can be outside of $[\mathrm{id}, \omega]$
- the flip order is not always the image of the weak order

Restrictions :

- only consider **strongly acyclic** pipe dreams
- order on pipe dreams : **acyclic order** (weaker than flip order)

Theorem

For any n-shape F and $\omega \in \mathfrak{S}_n$ sortable on F, the map $\mathrm{Ins}_{F,\omega}$ is a ***lattice morphism*** *from the* ***weak order interval*** $[\mathrm{id}, \omega]$ *to the* ***strongly acyclic*** *pipe dreams ordered by the acyclic order.*

Theorem

If the maximal permutation $\omega_0 = n\,(n-1)\,\dots 2\,1$ is sortable on F, then any linear extension of a pipe dream on F with exit permutation ω is in $[\mathrm{id}, \omega]$, and ***all acyclic pipe dreams are strongly acyclic****.*

Further generalization : Coxeter groups

symmetric group $\mathfrak{S}_n$	Coxeter group W
transpositions $(i, i+1)$	simple reflections
reduced pipe dreams	subword complex
pair of pipes	root in Φ
$P^\#$ acyclic	root cone is pointed
$\pi \in \mathrm{lin}(P)$	root configuration $\subseteq \pi(\Phi^+)$

Theorem

For any word Q on S and $w \in W$ sortable on Q, the map $\mathrm{Ins}_{Q,w}$ is ***well-defined*** *on the weak order interval $[e, w]$.*

Theorem (Jahn & Stump 2022)

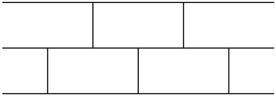
If the Demazure product of Q is w_0, then for any $w \in W$ the application $\mathrm{Ins}_Q(w, \cdot)$ is ***surjective on acyclic facets*** *of $\mathrm{SC}(Q, w)$.*

Conjecture

If Q is an alternating word on S and $w \in W$ is sortable on Q, then the application $\mathrm{Ins}_{Q,w} : [e, w] \mapsto \mathrm{SC}(Q, w)$ is a **lattice morphism** from the left weak order on $[e, w]$ to its image.

Thank you for your attention !

Q a word on S seen as a sorting network, here $\omega = \omega_0 = n(n-1)\dots 1$



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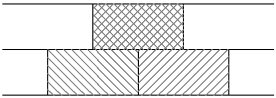
- **bricks** of Q : bounded cells



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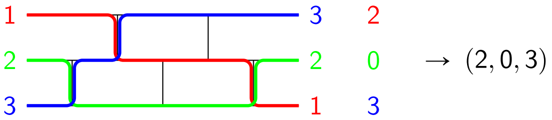
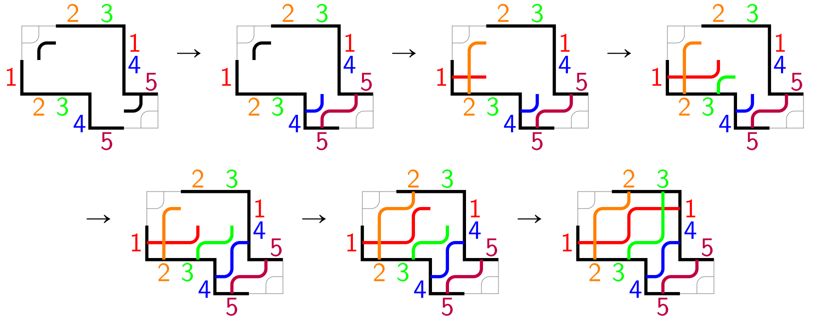
- **bricks** of Q : bounded cells
- **brick vector** of $f \in \mathrm{SC}(Q, \omega)$: i^{th} coordinate is the number of bricks under pipe i



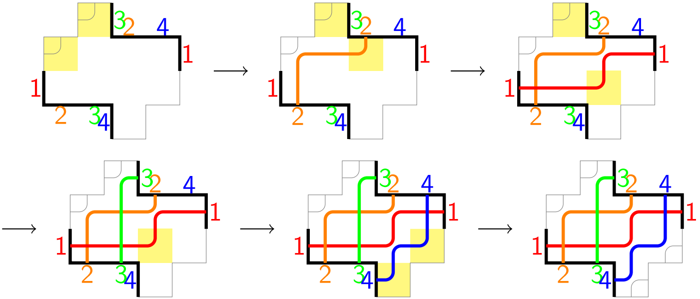
- **brick polytope** of $\mathrm{SC}(Q, \omega)$: convex hull of brick vectors of facets

Sweeping algorithm for $\omega = 23145$ and $\pi = 21345$

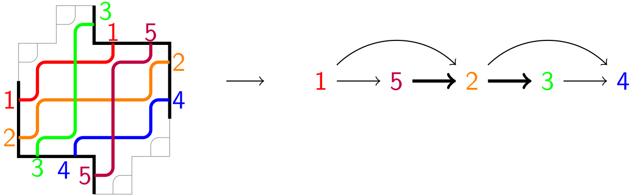


1. if $\omega^{-1}(i) < \omega^{-1}(j)$, add an elbow
2. if $\omega^{-1}(i) > \omega^{-1}(j)$ and $\pi^{-1}(i) > \pi^{-1}(j)$, add a cross
3. if i, j inversion of ω and non-inversion of π, add an elbow if you can still make the pipes end in order ω that way (3a), and a cross otherwise (3b)

Insertion algorithm for $\omega = 3241$ and $\pi = 2134$
The idea : keep track of cells containing only half of an elbow, and complete as many of those cells as possible when inserting a new pipe.



An acyclic but not strongly acyclic facet :



One linear extension : $15234 \not\leq 31524$.