Lattice properties of acyclic pipe dreams

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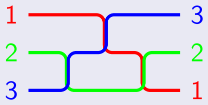
A generating set for $\mathfrak{S}_n$: $S = \{\tau_i = (i, i+1) \mid i \in [\![1, n-1]\!]\}$
$\to$ any permutation can be written as a word on the alphabet S.

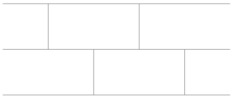
Example

$321 = \tau_2\tau_1\tau_2$
Graphical representation:

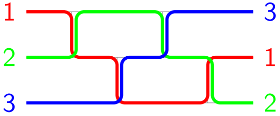


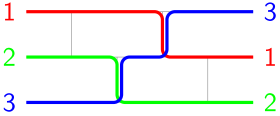
An idea: study the **subwords** of a word Q

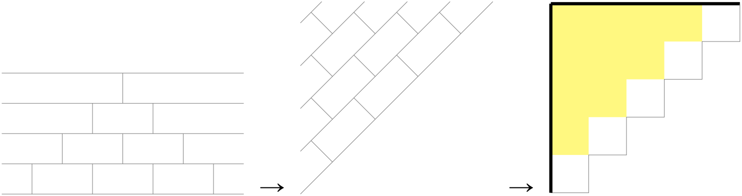
Word as a **sorting network**:

$Q = \tau_1\tau_2\tau_1\tau_2 \quad \rightarrow$ 

Representation of subwords:

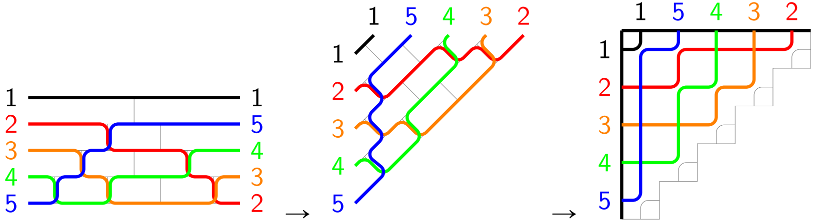
$\{1,2,3,4\}$: 

$\{2,3\}$: 



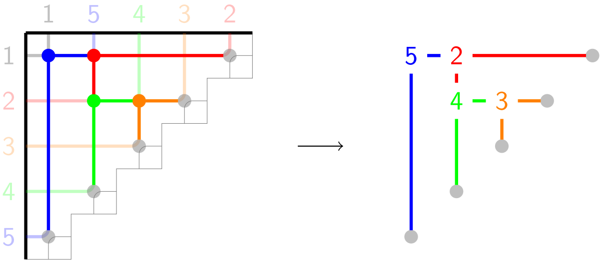
Pipe dream: filling the shape with crosses + and elbows ⌟⌜.

From sorting networks to pipe dreams



Set the exit permutation to $\omega = 1\, n\, (n-1) \ldots 2$
$\rightarrow$ every pipe except 1 has three elbows: two ⌟ and one ⌜.

What happens if we put vertices on the elbows and edges on the pipes between them?



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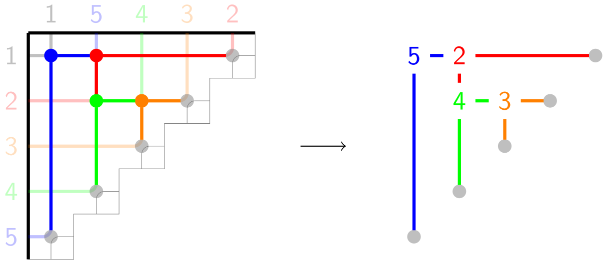
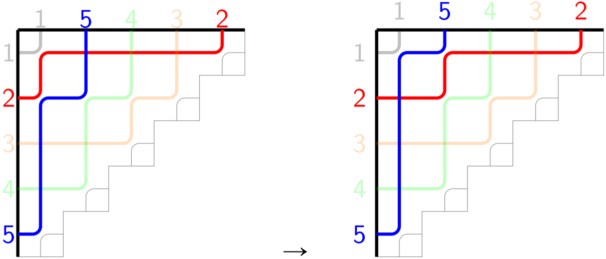


A (mirrored) binary search tree!

pipe dreams with n pipes $\longleftrightarrow$ binary trees with $n-1$ nodes

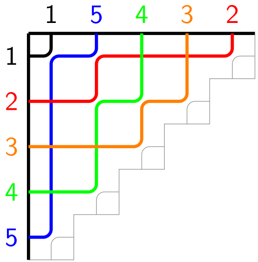
Equivalent to the rotation operation? To the insertion algorithm?
Yes and yes.

rotations $\longleftrightarrow$ flips (ascending here)



insertion algorithm $\longleftrightarrow$ drawing the pipes one by one

Ex: in order 15243

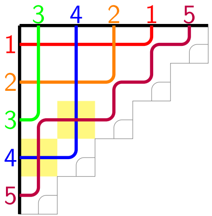


Previously: the set of pipe dreams with exit permutation $1\, n\, (n-1) \ldots 2$, partially ordered by oriented flip, is a lattice.

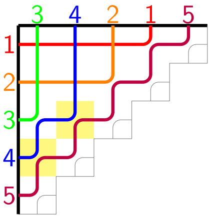
Can we extend this result?

First generalization: set any $\omega \in \mathfrak{S}_n$ as the exit permutation.
Problem: not all pipe dreams work.

They need to be **reduced**:



$\longrightarrow$



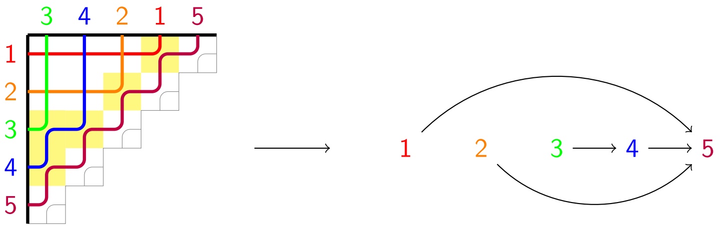
But it's still not enough! They also need to be **acyclic**.

Definition

P pipe dream, $P^{\#}$ the contact graph of P:

- vertex $\longleftrightarrow$ pipe
- edge from i to j $\longleftrightarrow$ contact

P acyclic $\iff$ $P^{\#}$ acyclic



$\Sigma(\omega)$: set of reduced acyclic pipe dreams of exit permutation ω

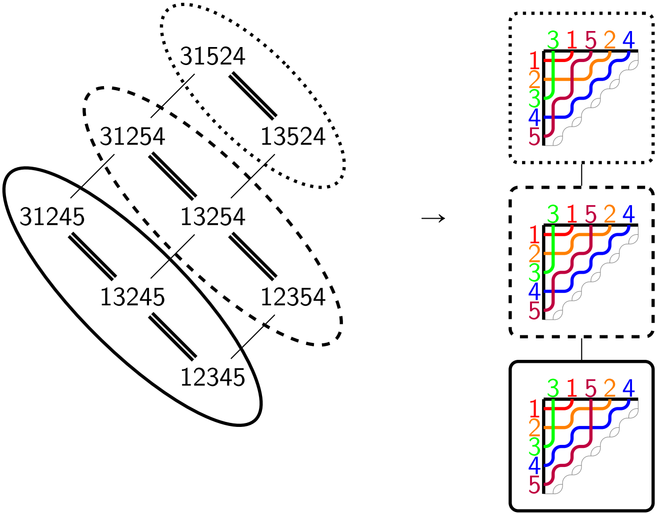
$\mathrm{lin}(P)$: **linear extensions** of $P^{\#}$

- $\{\mathrm{lin}(P)\}_{P \in \Sigma(\omega)}$ is a partition of the weak order interval $[\mathrm{id}, \omega]$
- $\mathrm{lin}(P)$ is an interval of the weak order
- the projections to the min and max of these intervals are order-preserving

Theorem (Pilaud)

The poset of acyclic reduced triangular pipe dreams of exit permutation ω is a lattice congruence of the weak order interval $[\mathrm{id}, \omega]$.

Ex: lattice congruence for $\omega = 31524$



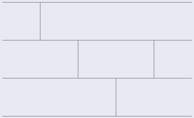
Can we generalize this result to other sorting networks?

❌ not true in most sorting networks, even ones sorting $n\,(n-1)\,\ldots 2\,1$

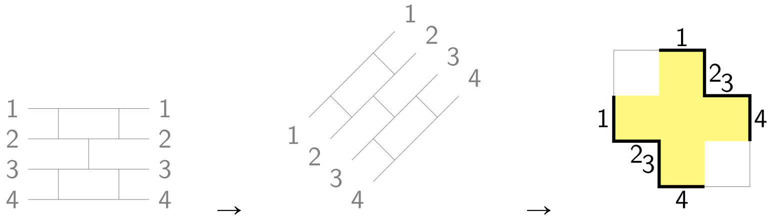
Restriction to **alternating** sorting networks:

Example

Alternating: 

Not alternating: 

An alternating sorting network on n pipes $\Longleftrightarrow$ an n-**shape**:

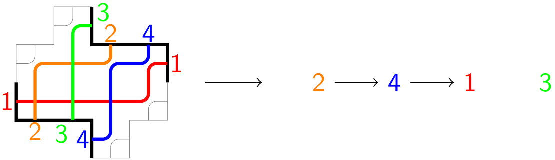


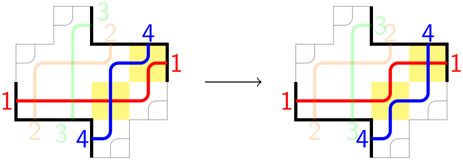
Structure of an n-shape:

- an entrance path of length n at its south-west
- an exit path of length n at its north-east
- a stair path at its north-west
- a stair path at its south-east

Pipe dream on an n-shape F: filling F with crosses ┼ and elbows ╯╭

Reduced pipe dreams, contact graphs, acyclic pipe dreams, flips: same



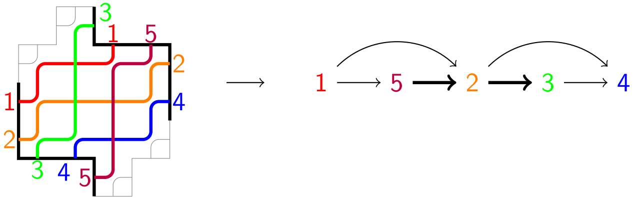


- $\Sigma_F(\omega)$: acyclic reduced pipe dreams on F with exit permutation ω
- $\mathrm{lin}(P)$: linear extensions of $P^{\#}$

$$\text{If } \Sigma_F(P) \neq \varnothing, \text{ is } \{\mathrm{lin}(P) \mid P \in \Sigma_F(\omega)\} \text{ a partition of } [\mathrm{id}, \omega]?$$

No.

A counter-example:



Some slightly weaker (but true!) properties:

- for $P, P' \in \Sigma_F(\omega)$, $P \neq P' \iff \mathrm{lin}(P) \cap \mathrm{lin}(P') = \varnothing$;
- $\bigcup_{P \in \Sigma_F(\omega)} \mathrm{lin}(P)$ is a lowerset;
- $[\mathrm{id}, \omega] \subseteq \bigcup_{P \in \Sigma_F(\omega)} \mathrm{lin}(P)$.

Lemma

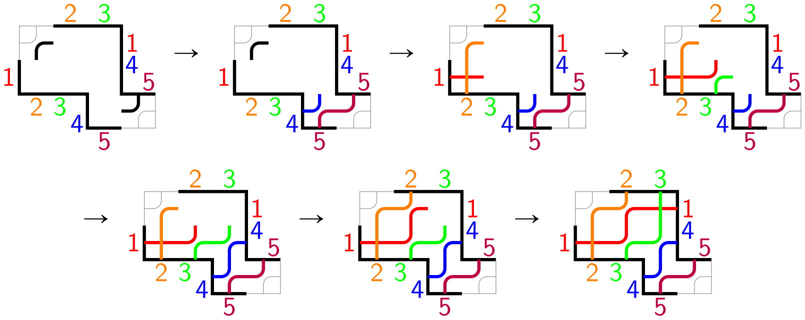
If $\pi = UbaV \in \mathfrak{S}_n$ is in $\mathrm{lin}(P)$ and $a < b$, then $\pi' = UabV$ is either in $\mathrm{lin}(P)$, or in $\mathrm{lin}(P')$ with P' obtained from P with a descending flip of pipes a and b.

$\text{Ins}_F(\omega, \pi)$: the only $P \in \Sigma_F(\omega)$ such that $\pi \in \text{lin}(P)$

How do we find $\text{Ins}_F(\omega, \pi)$?

Option 1: a **sweeping algorithm** to build P cell by cell.
Option 2: an **insertion algorithm** to build P pipe by pipe in order π.

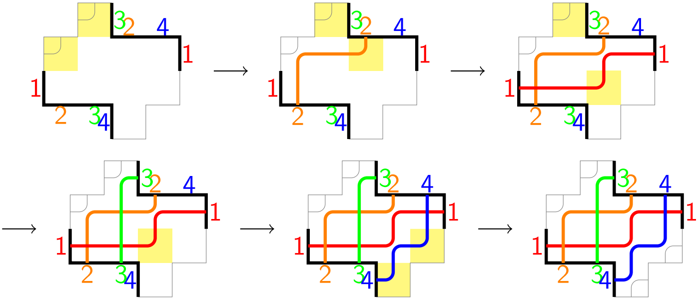
Sweeping algorithm: for $\omega = 23145$ and $\pi = 21345$



1. if $\omega^{-1}(i) < \omega^{-1}(j)$, add an elbow
2. if $\omega^{-1}(i) > \omega^{-1}(j)$ and $\pi^{-1}(i) > \pi^{-1}(j)$, add a cross
3. if i, j inversion of ω and non-inversion of π, add an elbow if you can still make the pipes end in order ω that way (3a), and a cross otherwise (3b)

The idea: keep track of cells containing only half of an elbow, and complete as many of those cells as possible when inserting a new pipe.

Example: $\omega = 3241$, $\pi = 2134$:



Does it work?

If $\pi \leqslant \omega$, yes, because:

- all free elbows accessible to a pipe can be completed at once;
- the pipes never go outside of the *n*-shape;
- if a pipe starts vertically (resp. ends horizontally), there will be an accessible free elbow facing its start (resp its end) when it is inserted;
- the resulting pipe dream is reduced.

The way $\mathrm{Ins}_F(\omega, \pi)$ is created guarantees that $\pi \in \mathrm{lin}(\mathrm{Ins}_F(\omega, \pi))$.

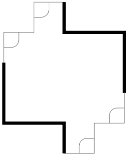
The insertion is order-preserving.

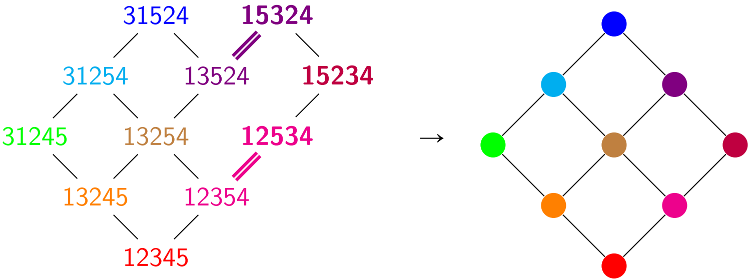
Is it a lattice morphism from $\bigcup_{P \in \Sigma_F(\omega)} \mathrm{lin}(P)$ to $\Sigma_F(\omega)$?

Not in general:

- $\bigcup_{P \in \Sigma_F(\omega)} \mathrm{lin}(P)$ is a lowerset but not always an interval;
- $\mathrm{Ins}_F(\omega, \cdot) : [\mathrm{id}, \omega] \mapsto \Sigma_F(\omega)$ is not always surjective.

But if $\Sigma'_F(\omega) = \mathrm{Ins}_F(\omega, [\mathrm{id}, \omega])$, then $\mathrm{Ins}(\omega, \cdot) : [\mathrm{id}, \omega] \mapsto \Sigma'_F(\omega)$ is a lattice morphism.

$\omega = 31524,\ F =$ 



Lemma

Let P be a reduced pipe dream on F and i, j be pipes.
If i has an elbow e weakly north-west to an elbow e' of j, and the rectangle with e as its north-west corner and e' as its south-west corner is contained in F, then $i \to j$ in $P^{\#}$.

Lemma

Let $P \in \Sigma_F(\omega)$ and $a, b, c \in [\![1, n]\!]$ such that $a < b < c$ and $\omega^{-1}(c) < \omega^{-1}(b) < \omega^{-1}(a)$.
If there is an elbow of P containing the pipes a and c, one of the two following statements is true:

- $a \to b$ *and* $c \to b$ *in* $P^{\#}$;
- $b \to a$ *and* $b \to c$ *in* $P^{\#}$.

Lemma (Björner & Wachs, 1991)

Let $\lhd$ be a partial order on $[\![1, n]\!]$. Suppose that for all $a < b < c$:

- $a \lhd c \Rightarrow a \lhd b$ *or* $b \lhd c$
- $c \lhd a \Rightarrow c \lhd b$ *or* $b \lhd a$

Then the set of linear extensions of $\lhd$ is an interval.

Theorem

For $P \in \Sigma_F(\omega)$, the set $\mathrm{lin}(P) \cap [\mathrm{id}, \omega]$ is an interval.

Lemma

Let $p^{\uparrow}$ *and* $p^{\downarrow}$ *be the respective projections on the top and the bottom of the fibers of* $\mathrm{Ins}_F(\omega, \cdot) : [\mathrm{id}, \omega] \mapsto \Sigma_F(\omega)$*. They are both order-preserving.*

Theorem

The map $\mathrm{Ins}_F(\omega, \cdot) : [\mathrm{id}, \omega] \mapsto \Sigma'_F(\omega)$ *is a lattice morphism.*

In which cases do we have $\bigcup_{P \in \Sigma_F(\omega)} \text{lin}(P) = [\text{id}, \omega]$ (and thus $\Sigma'_F(\omega) = \Sigma_F(\omega)$)?

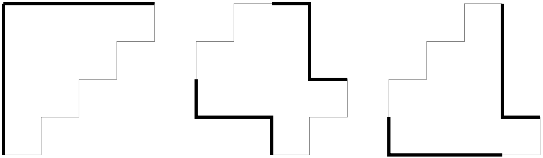
Theorem

If the alternating word corresponding to F contains a reduced expression of the permutation $n\,(n-1)\ldots 2\,1$, then for any $P \in \Sigma_F(\omega)$, $\text{lin}(P) \subseteq [\text{id}, \omega]$.

Thus in that case $\Sigma_F(\omega)$ is a lattice quotient of $[\text{id}, \omega]$.

Constructing an alternating reduced word of $n\,(n-1)\ldots 2\,1$:

- choose $\epsilon \in \{-,+\}^n$
- entrance path: from $(0,0)$, k^{th} step south if $\epsilon_k = -$, east if $\epsilon_k = +$
- exit path: from $(n,0)$, k^{th} step west if $\epsilon_k = -$, north if $\epsilon_k = +$
- stair paths from $(0,0)$ to $(|\epsilon|_+, |\epsilon|_+)$ and from $(|\epsilon|_+, -|\epsilon|_-)$ to $(n,0)$



$\epsilon = - - - -$ $\quad$ $\epsilon = - + + -$ $\quad$ $\epsilon = - + + +$

Can we generalize this even more?

symmetric group $\mathfrak{S}_n$	Coxeter group W
transpositions $(i, i+1)$	simple reflections
reduced pipe dreams	subword complex
pair of pipes	root in Φ
$P^{\#}$ acyclic	root cone is pointed
$\pi \in \mathrm{lin}(P)$	root configuration $\subseteq \pi(\Phi^+)$
sweeping algorithm	sweeping algorithm

Theorem

For $Q \in S^*$ and $w \in W$ such that $\mathrm{SC}(Q, w) \neq \emptyset$, and for $\pi \leqslant w$ in the weak order, $\exists! f \in \mathrm{SC}(Q, w)$ such that $\mathrm{Cone}(\mathbb{R}(f)) \subset \pi(\Phi^+)$.

Is the map $[e, w] \mapsto \mathrm{SC}(Q, w)$ a lattice morphism?

Experimentally, if Q is alternating, yes.

Is it surjective on acyclic subwords?

Experimentally, if $\mathrm{Demazure}(|Q|) = w_0$, yes.

Thank you for your attention.